Money, banking, and monetary policy

Ping He, Lixin Huang, Randall Wright

Tsinghua University, Beijing, PR China
Georgia State University, Atlanta, GA, USA
University of Pennsylvania, Philadelphia, PA 19104, USA

Abstract

An important function of banks is to issue liabilities, like demand deposits, that are relatively safe and liquid. We introduce a risk of theft and a safe-keeping role for banks into modern monetary theory. This provides a general equilibrium framework for analyzing banking in historical and contemporary contexts. The model can generate the concurrent circulation of cash and bank liabilities as media of exchange, or inside and outside money. It also yields novel policy implications. For example, negative nominal interest rates are feasible, and for some parameters optimal; for other parameters, strictly positive nominal rates are optimal.

© 2008 Elsevier B.V. All rights reserved.

Genuine banks are distinguished from other kinds of financial intermediaries by the readily transferable or ‘spendable’ nature of their IOUs, which allows those IOUs to serve as a means of exchange, that is, money. ... Commercial bank money today consists mainly of deposit balances that can be transferred either by means of paper orders known as checks or electronically using plastic ‘debit’ cards. George Selgin, Banking.

1. Introduction

Banks perform many functions in modern economies, but one very important function is to issue liabilities, like demand deposits, that are relatively safe and also liquid. Putting money in the bank obviously reduces the risk that it will get lost or stolen without excessively hindering its use as a means of payment. Moreover, using something other than cash reduces other risks, since one may be able to stop payment with a check or credit card, for example, if a purchase turns out to be flawed or fraudulent. While these points may be obvious, this does not mean they are uninteresting or unimportant for our understanding of money and banking. Yet they have been all but ignored in the literature.1

1 Gorton and Winton (2003)survey mainstream banking theory; it has nothing to say about media of exchange, let alone the relation between cash and bank liabilities in this role. Monetary theory along the lines of Kiyotaki and Wright (1989)determines media of exchange endogenously, but usually...
In a previous attempt to rectify this situation (He et al., 2005), we introduced a risk of theft into a micro-founded model of monetary exchange based on search theory. This allowed us to study the role of banks as institutions that provide safekeeping plus liquidity in a setting where there is an endogenous role for a means of payment in the first place. A drawback with that analysis, however, is that we used a rather crude physical environment. As in all simple first-generation search models, we adopted the assumption that money is indivisible and agents can only hold at most 1 unit. While this is unsatisfying for a number of reasons, perhaps the main limitation is that it is impossible to discuss many aspects of monetary policy, especially the effect of inflation or nominal interest rates on the use of currency and bank liabilities in payments.

The goal of this project is to continue the integration of banking and monetary theory by reconsidering these ideas in a more recent generation of search models where agents can hold any amount of money. This allows us to go well beyond the earlier work, especially concerning policy and the effects of inflation or nominal interest rates, and provides a general equilibrium framework in which to formalize venerable ideas about how banks evolved historically. Although we do not dwell on history here, it may be helpful to review the story told in standard reference books: “The direct ancestors of modern banks were ... the goldsmiths. At first the goldsmiths accepted deposits merely for safe keeping; but early in the 17th century their deposit receipts were circulating in place of money.” (Encyclopedia Britannica 1954, vol. 3, p. 41, emphasis added).2 This is precisely how banks operate and how consumers use them in the model.

While this history is fascinating, there are also contemporary issues for which our analysis is relevant. In terms of policy, an improved understanding of the design and implementation of modern payment systems may arise when we better understand simple situations like the one studied here. In terms of empirical work, research surveyed by Boyd and Champ (2003), for example, describes many findings concerning relations between inflation or interest rates and financial markets, including the banking sector. We do not attempt to address these observations directly, but we think our framework provides a step in the right direction, in that if one is to make sense of such empirical results, especially those concerning monetary policy, it might be useful to have a framework that better integrates banks and other financial institutions into monetary theory.

The rest of the paper is summarized as follows. In Section 2 we present basic assumptions. In Section 3 we study the case with exogenous risk of theft and no banks to show how the value of money depends on this risk. We show that it is possible in equilibrium to have negative nominal interest rates, although there is a lower bound. In fact, in this model it is optimal to go to the lower bound, which means deflation in excess of the Friedman Rule \( i = 0 \). In Section 4 we endogenize the risk associated with cash, still with no banks. In this version of the model, depending on parameters, it may or may not be possible to have negative nominal rates, but it will never be optimal: the optimal interest rate is either \( i = 0 \) or \( i > 0 \). The reason that some inflation in excess of the Friedman Rule may be optimal is that in equilibrium it reduces the risk associated with cash.

In Section 5 we introduce banks with exogenous theft. We show that generically agents either put all or none of their money in the bank, so we cannot get the concurrent circulation of multiple means of payment: bank liabilities drive cash out of circulation (or vice versa) when their operating costs are small (big). The optimal policy is \( i < 0 \) with banks and exogenous theft. In Section 6 we endogenize both theft and banking. Now we can generate concurrent circulation of multiple means of payment. We find in this version of the model the optimal policy is either \( i < 0 \) or \( i > 0 \). This is interesting because usually the Friedman Rule is extremely robust: \( i = 0 \) is optimal in a wide variety of models. In Section 7 we conclude.

Before proceeding, we comment further on the applicability of these ideas. The fact that mainstream banking theory mainly ignores payments and banks’ role in the provision of convenient, efficient, and safe instruments that facilitate this process might mean people who work in this tradition will not recognize many of the issues or the tools here, but this is no reason to dismiss the approach. In any monetary economy, or payment system, generally, safety is a real concern and

---

2 Similarly, “By the restoration of Charles II in 1660, London’s goldsmiths had emerged as a network of bankers… Some were little more than pawnbrokers while others were full service bankers. The story of their system, however, builds on the financial services goldsmiths offered as fractional reserve, note-issuing bankers. In the 17th century, notes, orders, and bills (collectively called demandable debt) acted as media of exchange that spared the costs of moving, protecting and assaying specie.” (Quinn, 1997, p. 411–12). “The crucial innovations in English banking history seem to have been mainly the work of the goldsmith bankers in the middle decades of the seventeenth century. They accepted deposits both on current and time accounts from merchants and landowners; they made loans and discounted bills; above all they learnt to issue promissory notes and made their deposits transferable by ‘drawn note’ or cheque.” (Joslin, 1954, p. 168). Safekeeping was also crucial for earlier episodes in banking history, going back to the Templars (Weatherford, 1997; Sanello, 2003), but goldsmiths seem to be the first bankers whose liabilities circulated as media of exchange. Previously, payments typically involved transferring funds from one account to another and “generally required the presence at the bank of both payer and payee” (Kohn, 1999).
significant resources are devoted to this end. It should be clear this does not just mean petty theft. Theft here is a way to capture formally fraud, embezzlement, counterfeiting and many other kinds of opportunistic behavior. It is also obvious that, as economists, we can only study tradeoffs between alternative payment instruments when they have different properties. We focus narrowly on the fact that some instruments, like cash, are low cost but risky; hopefully, the ideas and results are applicable to anyone interested generally in alternative payment instruments and the role of banks in the exchange system.\(^3\)

### 2. Basic assumptions

A \([0, 1]\) set of agents live in forever in discrete time. Borrowing from Lagos and Wright (2005), hereafter LW, each period agents meet in a centralized market (CM) and a decentralized market (DM). The CM is a frictionless market where consumption \(x\) and labor \(\ell\) (plus assets) are traded, and utility is quasi-linear: \(U(x) = \ell\), with \(U' > 0 \geq U''\) and \(U'(x^*) = 1\) for some \(x^* > 0\). The DM has different goods traded, and is characterized by frictions designed to generate a role for media of exchange. First, there is a double-coincidence problem, modeled by assuming agents trade bilaterally and, when \(i\) and \(j\) meet, the following is true: \(i\) wants something \(j\) can produce but not vice versa with probability \(\sigma \in (0, 1/2)\); \(j\) wants something \(i\) can produce but not vice versa with probability \(\sigma\); and neither wants what the other produces with probability \(1-2\sigma\).

If \(i\) produces \(q\) units of his output for \(j\) in the DM, the latter gets \(u(q)\) and the former gets \(-c(q)\), with \(u', c' > 0, u'' < 0, c'' > 0, u(0) = c(0) = 0\). Also, \(u(q^*) = c(q^*)\) for some \(q^* > 0\). Hence there are gains from DM trade. We want personal credit to be difficult, so \(i\) cannot simply offer \(j\) an IOU for \(q\), say, promising to pay in the next CM. The standard approach is to impose anonymity: \(j\) does not know who \(i\) is, and so \(i\) could renege on such a promise without the fear of repercussion, which makes a tangible medium of exchange essential.\(^4\) Of course, when we add banks, we have to be more careful about the notion of anonymity. Before doing so, however, we introduce fiat money as one candidate medium of exchange, and then add another friction: the possibility that it can be stolen in the DM.

The fact that cash can be stolen is the reason agents might deposit it in banks for safekeeping. What is a bank? At the abstract level of this analysis, it is an institution that accepts cash deposits in the CM, and promises to hold them until the next CM. In exchange it issues liabilities, like the receipts of the goldsmith bankers, or checkbooks and cash cards in more modern times. One can think of banks here as technologies for performing this operation, or as a set of agents (different from the consumers described above) with access to this technology—it does not matter for our purposes since competition will drive to zero any profit from engaging in this activity. Banks are assumed to honor their promises because of legal enforcement or, in a slightly extended version of the model, perhaps because of some alternative commitment device. Although highly stylized, we think this setup captures some salient features of banks.

We now say more about how agents can use bank liabilities for DM payments, when these agents are anonymous to a degree, as described above. The essential point is that we do not really need to have agents completely informationally isolated in the DM—we can allow them to observe something about each other, perhaps through the bank, but just not for free. Suppose I want to pay you using liabilities of my bank. For me to deposit my money in the bank in the first place, I may need some kind of relationship with my banker, including perhaps a record-keeping technology (although perhaps not—see footnote 5). One might also think that you only accept these liabilities from me if you have some sort of relationship with, or at least knowledge of, my bank. If you can communicate with my bank, you may be able to identify me, eliminating anonymity.

But if communication with these intermediaries is costly we may try to avoid it. Suppose I can use a bank card, for example, to transfer funds directly from my account to yours. One would not strictly speaking say trade is anonymous if this same technology allows you to learn about me, but it can be more convenient (use less resources) and potentially desirable for a variety of other reasons (including privacy) to transfer the funds without you learning my name, address, trading history, etc. This is precisely how debit and stored-value cards work. The pre-electronic incarnation was the paper check, which may have my name on it, but otherwise not give you a lot of information about me. If you know that I have money in the bank, which is automatic with debit and stored-value cards or certified checks, say, although it may have to be verified at a cost with personal checks, we can trade without you knowing much else.

Consider an American Express Travellers’ Check, which is about as liquid as cash, and very safe for two reasons: people are less interested in stealing it, because to pass it they have to match the signature; and even if lost or stolen American Express refunds your money. Travellers’ Checks allow you to make purchases while revealing essentially nothing about

---

\(^3\) We note that in principle there is no reason one could not integrate our approach with standard models of banks, including those that emphasize providing loans, pooling risk, monitoring projects, etc. Also, we acknowledge that one can always ask why banks are needed and governments themselves do not provide safer means of payment? Although this goes beyond the scope of the paper, one vision is that government should intervene only when necessary. They may have a role in the provision of currency, but if the private sector (competitive banking) does a good job providing safer alternatives, government need not interfere. This does not mean they are unconcerned with safety, of course, as big efforts are made, for example, to fight counterfeiting.

\(^4\) In the typical modern monetary model it is assumed that agents who meet in decentralized markets do not know anything about each other, including their names or their trading histories. See Kocherlakota (1998); Wallace (2001); Corbae et al. (2003); Araujo (2004) or Aliprantis et al. (2007) for formal discussions.
yourself to venders, even if they could in principle find out something from American Express. Because it avoids the cost of communicating with the intermediary, agents prefer to exchange the Travelers’ Check in what is basically an anonymous transaction—sellers see buyers’ signatures, verifying they did not steal the asset, but otherwise the former know nothing about the latter, and hence personal credit would not work. At the risk of belaboring the point, even if they could find out through the intermediary enough to make direct consumer loans, there are obvious advantages to simply transferring the asset.5

Currency accomplishes something similar—low-cost anonymous trade—but can be risky relative to other instruments. Of course, not all bank liabilities are perfectly safe: historically, bank notes were about as easy to steal as coins, and bank failures or robberies do occur. But it is obvious that carrying a Travelers’ check, certified check, or bank card is usually less risky than wandering around with a bundle of cash. To study the trade off between using cash and bank liabilities in payments we assume the latter are safe, but their use entails a fee \( \phi \), since record keeping or safety requires resources. This resource cost is \( a > 0 \) per dollar on deposit. Also, banks are required to keep a fraction \( \rho \) of deposits on reserve. As a benchmark we set \( \rho = 1 \), so banks keep all deposits in the vault, and zero profit implies \( \phi = a \).

Required reserves are a legal restriction, like the restriction that only government can issue fiat currency. These regulations are not innocuous: they impose a cost on banks from issuing liabilities, since these liabilities must be backed by reserves of government money. This cost depends on inflation, naturally. We impose required reserves for simplicity—there are many reasons for banks holding reserves other than legal restrictions (Cavalcanti et al., 1999; Lester, 2007) but we do not want to put everything in the model. Let \( M \) be the stock of money at the start of the CM, evolving over time according to \( \dot{M} = (1+\pi)M \), where \( \pi \) denotes any variable \( z \) next period. In steady state, \( \pi \) is the inflation rate, giving the cost of holding reserves. The government budget is \( \pi M + T = pG \), where \( T \) is a lump sum nominal tax and \( G \) is exogenous. This completes the basic model.

3. Exogenous theft

For now, a fixed fraction of the population \( \lambda \in (0,1) \) are thieves who try to rob you when you meet in the DM, and with probability \( \gamma \in (0,1) \) succeed. Thieves do not produce or consume anything in the DM, but act just like honest agents in the CM. We start with the case where the only asset is cash—there are no banks for now. Let \( W_j(m) \) and \( V_j(m) \) be the value functions for type \( j \) entering the CM and the DM with money \( m \), where \( j = t, h \) indicates a thief or an honest person. The CM problem is,

\[
W_j(m) = \max_{x,\ell} \{ U(x) - \ell + \delta V_j(\hat{m}) \} \quad \text{s.t.} \quad px = \ell + m - \hat{m} - T, \tag{1}
\]

where \( \hat{m} \) is the money taken out of the CM, \( w \) the nominal wage, \( p \) the price level, and \( \delta \) a discount factor between the CM and DM.

For simplicity, we set the real wage to 1 by assuming a linear technology, so \( w = p \). Then, assuming interiority and the second-order conditions hold, which can be guaranteed as in LW, we have the following. For a thief, who has no reason to keep or safety requires resources. This resource cost is \( a > 0 \) per dollar on deposit. Also, banks are required to keep a fraction \( \rho \) of deposits on reserve. As a benchmark we set \( \rho = 1 \), so banks keep all deposits in the vault, and zero profit implies \( \phi = a \).

Consider now the DM. For a thief, with \( m_i = 0 \),

\[
V_t(0) = \lambda \beta W_t(0) + (1 - \lambda) \gamma \beta W_t(\hat{m}) + (1 - \gamma) \beta W_t(0), \tag{2}
\]

where \( \beta \) is a discount factor between the DM and CM. Similarly, for an honest person with \( m \)

\[
V_h(m) = \lambda \gamma \beta W_h(0) + (1 - \gamma) \beta W_h(m) + (1 - \lambda) \sigma [u(q) + \beta W_h(m - d)] + (1 - \lambda)(1 - 2\sigma)\beta W_h(m). \tag{3}
\]

Since \( W_t \) is linear these actually simplify a lot: for example, \( V_t(0) = \beta W_t(0)(1 - \lambda)\gamma \beta \hat{m}/p \). This will be very useful below.

In (3), \( (q, d) \) are the terms of trade when the agent buys and \( (\bar{q}, \bar{d}) \) the terms when he sells, which could differ (off the equilibrium path) depending on the money holdings of the buyer and seller. We determine the terms of trade here using generalized Nash bargaining, with \( \theta > 0 \) denoting the bargaining power of the buyer. Given the buyer has \( m \) and the seller

---

5 In fact, notice with a Travelers’ Check or a stored-value card the issuing bank actually need know virtually nothing about the buyer. Suppose, for example, I give banker cash in exchange for a card with a PIN code. The card is a promise by the issuing agent, enforced legally, to deliver cash to any bearer with the correct PIN. To the issuer, for this transaction, I can be totally anonymous. I can transfer the right to said cash by giving you the card and PIN code (assumed verifiable). The card is safe: no one wants to steal it since it is worthless without the PIN (assuming thieves can pick one’s pocket but not force one to reveal the code). This is a logically consistent description of a physical environment, and a fairly realistic description of actual stored-value or debit cards, as well as Travelers Checks except that they use a signature instead of a PIN code.

6 The first term is his payoff to meeting an honest person, which occurs with probability \( 1 - \lambda \), in which case with probability \( \gamma \) he is successful and goes to the CM with \( \hat{m} \), while with probability \( 1 - \gamma \) he is unsuccessful and goes with 0.
where we obtain the order condition holds, such that

$$\{u(q) + \beta W_h(m - d) - \beta W_h(m)\}^d[\{c(q) + \beta W_h(m + d) - \beta W_h(m)\}]^{-\theta}$$

subject to the constraint $d \leq m$. Using $W'(m) = 1/p$, this reduces to

$$\max_{d \leq m}(u(q) - \beta d/p)^d[\{c(q) + \beta d/p\}]^{-\theta}. \tag{4}$$

This immediately implies the solution does not depend on $\bar{m}$ and depends on $m$ iff the constraint binds, and yields versions of standard results for models like the one in LW.

**Lemma 1.** Given the CM price level $p$, the solution to (4) is

$$q = \begin{cases} g^{\theta - 1}(\beta m/p) \text{ if } m < m^*, \\ g^* \text{ if } m \geq m^* \end{cases}$$

and $d = \begin{cases} \text{if } m < m^*, \\ m \text{ if } m \geq m^*. \end{cases}$

where $g^*$ solves $u'(q^*) = c'(q^*)$, $m^* = g(q^*)p/\beta$ and $g(\cdot)$ is given by

$$g(q) = \frac{\theta c(q)u'(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)}.$$  

**Proof.** It is easy to verify that the proposed solution maximizes the objective in (4) subject to $d \leq m$. □

**Lemma 2.** Let $L(q) = u'(q)/g(q) - 1$. Then

$$m > m^* \Rightarrow V'_h(m) = \frac{p}{\beta}(1 - \lambda'gamma)$$

$$m < m^* \Rightarrow V'_h(m) = \frac{p}{\beta}[1 - \lambda gamma + (1 - \lambda)sL(q)].$$

**Proof.** Use the implicit function theorem and Lemma 1. □

We will see $m < m^*$, so $\beta m/p = g(q)$, which is especially nice if $\theta = 1$ as then $g(q) = c(q)$. So Nash bargaining is very easy.\textsuperscript{7}

For the next result, let $\bar{y}$ be the nominal interest rate, defined by the Fisher equation $1 + \bar{y} = \beta p\delta h$. In models without theft, monetary equilibrium (with $q > 0$) exists iff $i > 0$. The next Lemma gives something quite different.\textsuperscript{8}

**Lemma 3.** A monetary equilibrium exists iff $i > i^* = -\lambda gamma$.

**Proof.** To show necessity, consider problem (1). The derivative of the objective function with respect to $\bar{m}$ is $\delta V'_h(\bar{m}) - 1/p$, where we obtain $V'_h(\bar{m})$ by updating $V'_h(m)$ in Lemma 2 one period. If $(1 - \lambda gamma)\delta p/\beta > 1/p$ then $\delta V'_h(\bar{m}) - 1/p > 0$ for all $\bar{m} > \bar{m}^*$ and the problem has no solution. We conclude that in any equilibrium we must have $\beta p\delta h > 1 - \lambda gamma$. This is equivalent to $i > i^*$.

To show sufficiency, we consider steady states where $q$ and $m/p$ are constant. Also, to ease the presentation we assume $\theta < 1$. Now for all $m > \bar{m}^*$, we have $\delta V'_h(\bar{m}) - 1/p < 0$ as long as $i > i^*$; and for all $m < \bar{m}^*$, we have in steady state

$$\delta V'_h(\bar{m}) - 1 = \frac{\delta p}{\beta[p][1 - \lambda gamma + (1 - \lambda)sL(q)] - 1/p}, \tag{5}$$

Straightforward analysis implies that for $\bar{m}$ near $\bar{m}^*$, the right-hand side of (5) is strictly negative. This means that the solution to the CM problem must be $\bar{m} < \bar{m}^*$. We are done if we can find an $m \in (0, \bar{m})$ solving the first-order condition $\delta V'_h(\bar{m}) - 1/p$, where the second-order condition holds.

For $m \in (0, \bar{m})$, the first-order condition can be rewritten using the Fisher equation and Lemma 2 as

$$L(q) = \frac{1 + \lambda gamma}{(1 - \lambda)s}.$$  

(6)

The second-order condition reduces to $V'_h < 0$, or equivalently $L'(q) < 0$, by Lemma 2. Now $L(0) = \infty$ under standard conditions, and $L(q^*) = 0$ by routine calculation. By continuity, there exists $q \in (0, q^*)$ satisfying (6), which means the first-order condition holds, such that $L(q) < 0$, which means the second-order condition holds. Once we have $q$, Lemma 1 gives $m = pg(q)/\beta$, where $m < m^*$ since $q < q^*$ (Lemma 1). Then $p = \beta m/g(q)$ is determined to clear the market with $\bar{m} = M(1 - \lambda gamma)$.

Other CM variables ($x_i, \ell_i$) are determined as above. This completes the construction of equilibrium. □

\textsuperscript{7} Still, we note there are closely related models with other bargaining solutions (Aruoba et al., 2007), price taking or posting (Rocheteau and Wright, 2005), and auctions (Galeniarios and Kircher, 2008, in press), all of which work as well. Consider, for example, price taking, and let $\bar{p}$ be the DM price of $q$, which generally differs from the CM price $p$. Assuming $\bar{p}$ is taken parametrically, sellers maximize $-c(q) + \beta p/p$ and buyers maximize $u(q) - \beta q/p$ s.t. $\bar{p}q \leq m$. The buyer’s constraint binds in equilibrium. Substituting this into the FOC from the seller’s problem, we obtain $\beta m/p = q c'(q)$. Thus, every result we derive also holds in the price-taking model if we replace $q(c(q))$ by $q c'(q)$.

\textsuperscript{8} For simplicity the proof of Lemma 3 assumes $\theta < 1$, but the result is also true at $\theta = 1$. The only difficulty is that while the limit as $m \uparrow \bar{m}^*$ of (5) is still negative, it is only strictly negative when $i > i^*$. If $\theta = 1$ and $i > i^*$ the CM problem is solved at $\bar{m} = \bar{m}^*$, but also at any $m > \bar{m}^*$ since (5) is 0 for all $m > \bar{m}^*$. As is standard, this indeterminacy is eliminated if we consider the limit as $i \uparrow i^*$. 
Notice $L(q)$ defined in Lemma 2 is a liquidity premium on cash, which (6) sets equal to the holding cost, given by the nominal rate $i$ plus the risk factor $\gamma$, times the expected number of periods until one spends the money $1/(1-\lambda)\sigma$. Because we know $L(q)<0$, $q$ is decreasing in $i$. Also, $q$ is increasing in $\theta$. We know $q<q^*$, with strict inequality except when $\theta = 1$ and $l = t^*$. Hence, we maximize expected utility by raising $q$ as high as possible. This means setting $i = t^*$, which implies $q = q^*$, where $L(q^*, i = 0)$, and $q^* < q^*$ with equality iff $\theta = 1$. Summarizing what has been established.

Proposition 1. With exogenous theft and no banks, there exists a monetary equilibrium iff $i > t^* = -\lambda \gamma$. In equilibrium $\partial q/\partial i < 0$, the optimal policy is $i = t^*$, and it implies $q = q^*$, where $q^* < q^*$ with equality iff $\theta = 1$.

The fact that $i$ can be negative is unusual, but not surprising given $m$ is risky. The usual arbitrage argument to rule out $i < 0$ is that one could borrow a dollar today, pay back $1+i$ tomorrow, and have money left over. But now the dollar might get stolen! Arbitrage now says $i$ cannot be too negative. Empirically, $i < 0$ is actually quite common, as evidenced by Travelers’ checks and demand deposits with service fees, so it is good to have a model consistent with this observation.

4. Endogenous theft

We now endogenize the number of thieves, and hence the risk of using cash. Clearly we cannot have $\lambda = 1$, since no one will work to acquire $m$ when no one else is honest. So we look for equilibria with $\lambda \in (0,1)$. To this end, let $\Delta(m) = W_w(m) - W_h(m)$, and note that $\Delta$ actually does not depend on $m$, since both $W_h$ and $W_w$ are linear with slope $1/p$. Then equilibrium requires, in addition to the conditions discussed in Section 3,

\[ \lambda = 0 \text{ if } \lambda = 0, \text{ and } A = 0 \text{ if } \lambda \in (0,1). \] (7)

From the previous section, we can write payoffs in the DM as

\[ V_t(0) = \beta W_t(0) + \beta(1-\lambda)\gamma m/p, \]
\[ V_h(m) = \beta W_h(0) + (1-\lambda)\sigma[u(q) - c(q)] + \beta(1-\lambda)\gamma m/p. \]

Similarly, in the CM,

\[ W_t(m) = U(x^*) - x^* - T/p + m + \delta V_t(0), \]
\[ W_h(m) = U(x^*) - x^* - T/p + m + \delta V_h(m) - \bar{m} - \bar{m}. \]

Updating $V_t$ and $V_h$ one period, inserting these into $W_t$ and $W_h$, and imposing steady state, we have

\[ (1 - \delta\beta)W_t(0) = U(x^*) - x^* - T/p + \delta(1-\lambda)\gamma \bar{m}/p, \]
\[ (1 - \delta\beta)W_h(0) = U(x^*) - x^* - T/p + \delta(1-\lambda)\sigma[u(q) - c(q)] + \delta(1-\lambda)\gamma \bar{m}/p + \bar{m}/p. \]

Using these, as well as the bargaining solution $\beta m/p = g(q)$ and the Fisher equation, we can simplify $\Delta$ to

\[ \Delta \approx (1-\lambda)\sigma[u(q) - c(q)] - (i + \gamma)g(q), \] (8)

where the notation $\Delta \approx B$ means $A$ and $B$ have the same sign. There are two possible types of equilibria. One possibility is $\lambda \in (0,1)$, which requires $\Delta = 0$, and therefore

\[ \lambda = 1 - \frac{(i + \gamma)g(q)}{\sigma[u(q) - c(q)]}. \] (9)

The other possibility is $\lambda = 0$, which requires $\Delta > 0$. In either case we still have to satisfy equilibrium condition (6) for $q$ from Section 3.

To fix ideas, consider an example with $\theta = 1$, which means $g(q) = c(q)$, and functional forms $u(q) = q^x$ and $c(q) = q$ (general results are given below). Consider equilibria with $\lambda > 0$. Then (9) and (6) imply

\[ \lambda = \frac{x(1-\lambda)\gamma}{\gamma - \sigma(1-x)}. \] (10)

---

9 In our model agents always evaluate welfare the same way. In the endogenous theft case, thieves and honest agents actually get the same payoff; in the exogenous theft case, they do not get the same payoff, but agree on the best level of $q$ and hence the optimal policy.

10 Because we assumed homogeneous agents, if we are to have an interior solution $0 < \lambda < 1$, they will be indifferent between being thieves and being honest in equilibrium. Hopefully it is understood, however, that everything goes through if agents were heterogeneous with respect to some characteristic $k$, say, the cost of becoming a thief, except that equilibrium determines a $\lambda^*$ consistent with indifference while anyone with $k \neq k^*$ strictly prefers either being a thief or being honest.
The solution of $i = \frac{\lambda}{\lambda} \frac{1}{\lambda}$ is independent of $i$. Also, $\lambda > 0$ if $i < i_0 \equiv \alpha/(1 - \lambda)$, and $\lambda < 1$ if $i > -\gamma$ which is not binding since $i > i^* = -\gamma/\lambda$ (Lemma 3 applies). Hence, equilibrium with $\lambda \in (0,1)$ exists iff $i < i_0$. Now consider equilibria with $\lambda = 0$, which means

$$q = \left[ \frac{\alpha}{\sigma + 1} \right]^{1/1-\lambda}.$$  

This equilibrium requires $\lambda > 0$, which holds iff $i \geq i_0$. Now the solution of $q$ in (12), denoted as $q_0$, is decreasing in $i$ in equilibrium.

Summarizing, we have two generic cases.

Case (i) $\sigma(1-\lambda) > \alpha/\gamma$: Then $i_0 < 0$, and equilibrium exists iff $i > i^* = 0$. In this equilibrium $\lambda = 0$, and (12) gives $q = q_0$ as a decreasing function of $i$. Given $\theta = 1$, in this example, we get $q_0 = q^* = x^{1/(1-\lambda)}$ at $i = 0$; more generally we get $q_0 = q_\lambda$ at $i = 0$, where $q_\lambda$ is the solution to (6), that is, $L(q_\lambda) = 0$.

Case (ii) $\sigma(1-\lambda) < \alpha/\gamma$: Then $i_0 > 0$. An equilibrium with $\lambda = 0$ and $q_0$ given by (12) exists iff $i \geq i_0$, and an equilibrium with $\lambda > 0$ and $q_\lambda$ given by (11) exists iff $i \in (i^*, i_0)$. Now $i^* = -\gamma/\lambda$ is endogenous, but it is easy to check that $\lambda \uparrow 1$ as $i \downarrow i^*$, thus we must have $i^* = -\gamma$. Since $q_0$ is independent of $i$ when $\lambda > 0$, (10) implies that $\lambda$ is linearly decreasing in $i$ in this equilibrium.

Fig. 1 depicts case (ii). The curve for $q$ shows $q = q_0$ from (12) for $i > i_0$ and $q = q_\lambda$ from (11) for $i < i_0$. The curve for $\lambda$ comes from (9). Notice that as $i$ falls below $i_0$ we do not increase $q$, but simply increase $\lambda$ and hence the risk associated with money. Therefore, decreasing $i$ below $i_0$ clearly lowers welfare because it hurts trade at the extensive margin, by reducing the number of honest agents, with no change at the intensive margin, given $q_\lambda$, does not vary when $i \in (i^*, i_0)$. Hence, optimal policy is $i = i_0$.

We now show that these results actually do not depend on the parametric specification at all. In general, if $\lambda > 0$ then we can solve (9) for $\lambda$ and insert it into (6) to derive

$$L(q) = \frac{\sigma[u(q) - c(q)] - \gamma g(q)}{\sigma g(q)},$$

as long as $i > -\gamma/\lambda^*$ (since we divide by $i + \gamma$ to get this). At $i = i^*$, we have $\lambda = 1$ and $q = q_\lambda$ where $L(q_\lambda) = 0$. We can show the solution $q_\lambda$ to (13) does not depend on $i$, and so $\lambda$ is linearly decreasing in $i$ by (9). There is a value for $i$ given by

$$i_0 = \frac{\sigma[u(q_\lambda) - c(q_\lambda)]}{\gamma g(q_\lambda)} - \gamma$$

such that at $i_0$ we have $\lambda = 0$.

We know from the example that $i_0$ can be positive or negative. If $i_0 < 0$ then $i^* = 0$, equilibrium exists iff $i \geq 0$, and in equilibrium $\lambda = 0$. And if $i_0 > 0$, the situation is qualitatively exactly as in Fig. 1. Summarizing:

**Proposition 2.** With endogenous theft and no banks, there is a value of $i_0$, which can be positive or negative, with the following properties. If $i_0 < 0$ then equilibrium exists if $i \geq 0$, and it implies $\lambda = 0$ and $\gamma q/\gamma i < 0$. In this case welfare is maximized at $i = 0$. 

---

\footnote{If we could somehow increase $\sigma$ to make honest economic activity more attractive, or decrease $\gamma$ to make criminal activity less attractive, $\lambda$ falls and $i_0$ falls, allowing us to further decrease $i$ and increase welfare.}
If \( i_0 > 0 \) then equilibrium exists with \( \lambda = 0 \) if \( i \geq i_0 \) and it implies \( \hat{q}/\hat{i} < 0 \), while equilibrium exists with \( \lambda > 0 \) if \( i^* \leq i < i_0 \) where \( n^* = -\gamma \), and it implies \( q = q_0 \), independent of \( i \). In this case, welfare is maximized at \( i = i_0 \).

Given that the optimality of the Friedman Rule \( i = 0 \) is extremely robust in monetary economics, perhaps the most interesting part of these results is that welfare can be maximized at \( i = i_0 > 0 \). Now, in the previous section we found it was optimal to have \( i = i^* < 0 \), but this is in the spirit of the Friedman Rule, in the sense that it says to drive interest rates as low as possible. Here it can be optimal to have \( i \) above the lower bound. The reason is that it keeps people honest—that is, it reduces \( \lambda \).

5. Banks and exogenous theft

We now allow agents in the CM to put money in the bank.\(^{12}\) Here we assume the required reserve ratio is \( \rho = 1 \), so with a resource cost of a dollars per dollar on deposit the equilibrium fee is \( \phi = a \). Clearly, thieves do not use banks, so their CM problem is unchanged, while their DM payoff is

\[
V_l(0) = \lambda \beta W_l(0) + (1 - \lambda)(1 - \gamma) \beta W_l(m - b - m) + (1 - \gamma) \beta W_l(0)
\]
given the representative honest agent now carries \((1 - b)m\) in cash and has \(b \hat{m} \) in the bank.

An honest person’s CM problem becomes

\[
W_h(m) = \max_{x, \hat{m}, b} \left\{ U(x) - x + \frac{m - T - \hat{m}}{p} + \delta V_h(\hat{m}, \hat{b}) \right\}
\]

where \( m \) here is money left over after paying \( \phi \) to the bank. The bargaining solution is still given by Lemma 1 and the DM payoff by

\[
V_h(m, b) = \lambda \beta W_h(bm - bma) + (1 - \gamma) \beta W_h(m - b - m) + (1 - \lambda) \beta \sigma [u(q) + \beta W_h(m - d - bma)]
\]

\[+ (1 - \lambda) \sigma [-c(q) + \beta W_h(m + \hat{d} - bma)] + (1 - \lambda)(1 - 2\sigma) \beta W_h(m - bma).\]

Differentiating with respect to \( m \), we obtain the generalized version of Lemma 2

\[
m > m^* \Rightarrow V_{hm} = \frac{\beta}{p} [1 - \lambda \gamma (1 - b) - ab],
\]

\[
m < m^* \Rightarrow V_{hm} = \frac{\beta}{p} [1 - \lambda \gamma (1 - b) - ab - (1 - \lambda) \sigma L(q)].
\]

The obvious generalization of Lemma 3 now implies monetary equilibrium exists if \( i \geq i^* = -\lambda \gamma (1 - b) - ab \).

Given \( i > i^* \), the first-order condition for \( W_h(m) \) with respect to \( \hat{m} \) implies

\[
L(q) = \frac{i + \lambda \gamma (1 - b) + ab}{(1 - \lambda) \sigma},
\]

which generalizes (6). To determine \( b \), notice that the first order condition with respect to \( b \) satisfies

\[
V_{hb} \simeq \gamma \lambda - a.
\]

Hence, \( b = 1 \) if \( \gamma \lambda > a \) and \( b = 0 \) if \( \gamma \lambda < a \). Except in the non-generic case \( \lambda \gamma = a \); therefore, we cannot get concurrent circulation of cash and bank liabilities in general. In particular, for small \( a \) all the cash is in bank vaults, and all DM payments are made using liabilities—a “cashless economy.” However, as we will see in the next section, this result hinges on there being nothing to adjust to get \( b \in (0, 1) \) endogenously. First we summarize the above results.

**Proposition 3.** With exogenous theft and banks, if \( \lambda \gamma > a \) then \( b = 1 \), and if \( \lambda \gamma < a \) then \( b = 0 \). In either case, equilibrium exists if \( i \geq i^* = -\lambda \gamma (1 - b) - ab \), and it implies \( \hat{q}/\hat{i} < 0 \). The optimal policy is \( i = i^* \).

6. Banks and endogenous theft

Generalizing the expressions in Section 4, we have

\[
(1 - \delta \beta) W_l(0) = U(x^*) - x^* - \frac{T}{p} + \delta \beta (1 - \lambda) \gamma (1 - b) \hat{m} \frac{\hat{m}}{p},
\]

\[
(1 - \delta \beta) W_h(0) = U(x^*) - x^* - \frac{T}{p} + \delta (1 - \lambda) \sigma [u(q) - c(q)] + \delta \beta [1 - \lambda \gamma (1 - b) - ab] \hat{m} \frac{\hat{m}}{p} - \hat{m} \frac{\hat{m}}{p}.
\]

\(^{12}\) This model is different from He et al. (2005), which has indivisible money. There some agents put money in the bank, others do not, and equilibrium determines the fraction of each. Here all agents put a fraction \( b \) of their money in the bank.
Thus, the analog of (8) is

$$\Delta \equiv (1 - \lambda)\sigma[u(q) - c(q)] - [i + \gamma(1 - b) + abg(q)].$$

There are three possibilities for equilibrium: $b = 1, b = 0$, and $b \in (0,1)$. We cannot have $b = 1$ as long as $a > 0$: if $b = 1$ then no one holds cash, which leads to $\lambda = 0$, which means $b = 1$ cannot be a best response. Hence we restrict attention to $b \in (0,1)$.

Consider equilibrium with $b = 0$, which is a best response iff $\lambda \gamma \leq a$, by (16). In this case we determine $q$ and $\lambda$ exactly as in Section 4. In particular, to repeat the relevant parts of Proposition 2: $\lambda \in (0,1)$ implies $q = q_\Phi$ is independent of $i$; $\lambda \in (0,1)$ iff $i_0 > 0$ and $i < i_0$; if $i_0 < 0$ then $i^* = 0$ and $\lambda = 0$; and if $i_0 > 0$ then $i^* = -\gamma$, in which case $\lambda > 0$ for $i \in [i^*, i_0)$ and $\lambda = 0$ for $i \geq i_0$.

Given this, we check the best response condition. If $\lambda = 0$ then $\lambda \gamma \leq a$, and $b = 0$ is obviously a best response. The only nontrivial case is $i_0 > 0$ and $i \in [i^*, i_0)$, where $\lambda \in (0,1)$ is given by (9). In this case $\lambda \gamma \leq a$ iff $i \geq i_1$, where

$$i_1 = \frac{(\gamma - a)\sigma[u(q_\Phi) - c(q_\Phi)]}{\gamma g(q_\Phi)} - \gamma = \left(1 - \frac{a}{\gamma}\right)i_0 - a.$$

So when $i_0 > 0$ and $i \in [i^*, i_0)$, $b = 0$ is an equilibrium iff $i \geq i_1$.

This is shown in Fig. 2, where as $i$ decreases from $i_0$ towards $i_1$, $q = q_\Phi$ stays fixed and $\lambda$ increases from 0 to $a/\gamma$, as in Fig. 2. At $i = i_1$, however, something interesting happens. If $i$ decreases further, we get $\lambda > a$ and the best response becomes $b = 1$. But $b = 1$ cannot be an equilibrium, so what must happen is the following: when $i$ falls below $i_1$, we stay at $\lambda = a/\gamma$ so that $b \in (0,1)$ is a best response; and $b$ adjusts so that $\lambda = a/\gamma$ is a best response. Hence, equilibrium for $i \in [i^*, i_0)$ is determined by three conditions: (i) $b \in (0,1)$ is a best response, which means $\lambda = a/\gamma$; (ii) $\lambda \in (0,1)$ is a best response, which means $A = 0$, or

$$\lambda = 1 - \frac{i + ab + \gamma(1 - b)g(q)}{\sigma[u(q) - c(q)]};$$

and (iii) $q$ satisfies the usual condition (15).

When $b \in (0,1)$, DM transactions are made using both bank deposits and money.\footnote{In fact, each buyer makes every purchase using both, but this stands in for the idea that agents make some purchases with one and some with the other. Various extensions can be used to capture this formally in the model.} To discuss it further, assume $a < \gamma$ (because $0 < b < 1$ requires $\lambda = a/\gamma < 1$). Since $\Gamma = -a$, in this case, (18) implies $i_1 \in (i^*, i_0)$. Now, given $i \in [i^*, i_0)$, this equilibrium exists, and has a nice recursive structure: first set $\lambda = a/\gamma$; then (15) reduces to

$$L(q) = \frac{\gamma(i + a)}{\sigma(\gamma - a)}.$$
which can be solved for \( q = q_* \) with \( \partial q_* / \partial i < 0 \); then finally (19) can be solved for
\[
 b_* = \frac{i + \gamma}{\gamma - a} - \frac{\sigma[u(q_*) - c(q_*)]}{\partial g(q_*)/\partial q}.
\] (21)

Notice \( b_* = 1 - \sigma[u(q_*) - c(q_*)]/\gamma g(q_*) < 1 \) at \( i^* \) and \( b_* = 0 \) at \( i = i_1 \).

Summarizing what we have shown:

**Proposition 4.** With endogenous theft and banks, there is an \( i_0 \) with the following properties. If \( i_0 < 0 \) then monetary equilibrium exists iff \( i \geq 0 \), and it implies \( \lambda = b = 0 \) and \( \partial q / \partial i < 0 \). If \( i_0 > 0 \) then there is an \( i_1 \triangleq (i^*,i_0) \), where \( i^* = -a \) in this case, such that the following is true. Equilibrium with \( \lambda = b = 0 \) exists iff \( i > i_0 \) and it implies \( \partial q / \partial i < 0 \); equilibrium with \( \lambda \in (0,1) \) and \( b = 0 \) exists iff \( i \in (i_1,i_0) \) and it implies \( q = q_0 \) independent of \( i \) and \( \partial \lambda / \partial i < 0 \); and equilibrium with \( \lambda \in (0,1) \) and \( b \in (0,1) \) exists iff \( i \in (i^*,i_1) \) and it implies \( \partial q / \partial i < 0 \) and \( \lambda = a/\gamma \) independent of \( i \).

Finally, we turn to welfare. When \( i_0 \leq 0 \) we have \( i^* = 0 \) and \( \lambda = 0 \); in this case there is no role for banks and \( i = 0 \) is optimal. We therefore focus on \( i_0 > 0 \) from now on, as in Fig. 2. Consider first maximizing welfare over the range \([i_1,\infty)\). As in Section 4, the solution is \( i_0 \), since \( q \) is constant and \( \lambda \) decreasing over \([i_0,\infty)\), while in \([i_0,\infty)\), \( \lambda \) is constant and \( q \) decreasing. Now consider maximizing over \([i^*,i_1]\). In this range, \( \lambda \) is constant while \( q \) and \( b_* \) depend on \( i \), and since banking is costly, the optimal policy is not obvious and we must calculate welfare explicitly. Our criterion \( W \) is the CM payoff of an agent holding \( M \) (note that when \( \lambda \) is endogenous all agents get the same utility conditional on \( m \)).

For equilibrium with \( \lambda, b \in (0,1) \), we show in the Appendix A that
\[
 W = \left(1 - \frac{a}{\gamma}\right) \left\{ \sigma[u(q) - c(q)] - ag(q) \right\} \frac{i + \gamma}{\gamma - a}.
\] (22)

The first term in braces gives the expected surplus from DM trade, while the second term gives the endogenous resources used up by banking activity. Similarly, with \( b = 0 \) and \( \lambda \in (0,1) \) we show
\[
 W = \frac{\left( (i + \gamma)g(q) \right)^2}{\sigma[u(q) - c(q)]},
\] (23)

and with \( b = \lambda = 0 \) we show
\[
 W = \sigma[u(q) - c(q)].
\] (24)

In Fig. 3 the upper curve corresponds to strategies \( b = \lambda = 0 \), the lower curve to \( \lambda, b \in (0,1) \), and the middle curve to \( b = 0 \) and \( \lambda \in (0,1) \).

It is clear \( W \) is either maximized globally at \( i = i_0 \), or at some point \( i \in (i^*,i_1] \). We show in the Appendix A that the maximum over \([i^*,i_1]\) occurs at \( i < 0 \), but it could be either \( i = i^* \) or \( i \in (i^*,0) \). In Fig. 3, the global max occurs at \( i \in (i^*,0) \), but it can occur also at \( i_0 \). The big observation is that we never want to run the Friedman Rule, but either \( i < 0 \) or \( i > 0 \), depending on parameters. The trade off is that at \( i_0 > 0 \), \( q \) is very low, but at least it eliminates criminals and bankers. Moving outside the confines of the formal setup, the general idea is that low inflation may encourage undesirable behavior (in the model, crime), which is not only unproductive but diverts resources in response (here, banking).

**Proposition 5.** With endogenous theft and banks, if \( i_0 < 0 \) then the optimal policy is \( i = 0 \), and if \( i_0 > 0 \) then the optimal policy may be either \( i \in (i^*,0) \) or \( i = i_0 > 0 \).
7. Conclusion

We studied models where as a medium of exchange agents may use cash, bank liabilities, or both. Basically, the advantage of currency is that it can be exchanged at low cost in situations where agents have little knowledge of each other. This leads to a disadvantage—currency can be stolen. Alternative means of payment, modeled here as bank liabilities, mitigate the theft problem, but systems with these alternatives are costly to operate. Our theory differs from the mainstream banking literature by emphasizing the role of banks and their liabilities in payments. The model generates novel policy predictions. It is feasible to have $i < 0$, and for some parameters this is optimal. For other parameters it is optimal to have $i > 0$.

Our setup is simplistic, but one could combine it with other banking models, and alternative environments could be considered. One could, for example, try to replace theft by private information, as in the monetary models of Williamson and Wright (1994), Trejos (1997) or Berentsen and Rocheteau (2004). Other extensions include reducing the reserve ratio below $\rho = 1$ and deriving a money multiplier, as in our earlier paper. One can also think of $\rho$ as a policy tool. With $\rho < 1$ we obtain $\phi < a$, since banks earn revenue from loans as well as fees, and we obtain $\phi < 0$ (interest on checking account) if $a$ and $\rho$ are low. In this case $b = 1$, since deposits are equally liquid, have a higher yield, and are safer than money. While this captures a "cashless economy" one could add features to make deposits less liquid—some agents do not accept checks, say. All this is left for future research.

Appendix A

Here we verify some claims made in Section 6. First we derive the expressions for welfare. In equilibrium with $\lambda, b \in (0,1)$, we have $W_0(M) = W_\lambda(M)$, and we compute

$$1 - \delta b W_t(M) = (1 - \delta b)W_t(0) + (1 - \delta b)\frac{M}{p} = U(x^*) - x^* + \delta b(1 - \lambda)\gamma(1 - b)\frac{\bar{m}}{\bar{p}} + (1 - \delta b)\frac{m(1 - \lambda)}{p}$$

$$= U(x^*) - x^* + \frac{M}{p}[\delta \beta \gamma(1 - b) + 1 - \delta \beta + \pi]$$

using $T = -\pi M, M = m(1 - \lambda), m/p = \bar{m}/\bar{\beta}$.

Using $m/p = g(q)/\beta$ and the Fisher equation, this becomes

$$(1 - \delta b)W_t(M) = U(x^*) - x^* + \delta(1 - \lambda)g(q)[\gamma i + \gamma(1 - b)].$$

After inserting $\lambda = a/\gamma$, eliminating $b$ using (21), and performing routine simplifications, we arrive at

$$(1 - \delta b)W_t(M) = U(x^*) - x^* + \delta \left(1 - \frac{a}{\gamma}\right)\left\{\sigma[u(q) - c(q)] - a g(q)\frac{i + \gamma}{\gamma - a}\right\}.$$

In the text, we use $W = [(1 - \delta b)W_t(M) x^* - U(x^*)]/\delta$ since we can neglect constants. This yields (22)–(24) are similar.

We now verify the optimal policy over $[i^*, i_1]$ is $i < 0$. Suppose we maximize (22) over $[i^*, i_1]$ by choosing $(q, i)$ subject to (20). Using the constraint to eliminate $i$, the problem becomes

$$\max_q W = \left(1 - \frac{a}{\gamma}\right)\left\{\sigma[u(q) - c(q)] - a g(q)\frac{i + \gamma}{\gamma - a}\right\}.$$

Differentiating with respect to $q$, with some manipulation, we obtain

$$\frac{\partial W}{\partial q} \equiv \sigma(u' - c') - a\left[\sigma L(q) + 1\right]g' - a\sigma L'(u' - g') - ag' - a\sigma L' \equiv \sigma(g' - c') + ig' - ag \frac{\sigma}{\gamma} L'. $$

One can show $g' > c$ and $L' < 0$ in equilibrium. Hence, $i \geq 0$ implies $\partial W/\partial q > 0$, and therefore $i \geq 0$ implies $\partial W/\partial i < 0$.

Finally, we verify that we cannot say generally whether the optimal policy over $[i^*, i_1]$ is $i^*$ or $i > i^*$. Consider the example with $\theta = 1, c(q) = q$ and $u(q) = q^2$. After computing equilibrium explicitly, we have

$$\frac{\partial W}{\partial q} \equiv i - a\frac{\sigma}{\gamma}(x - 1)\left[1 + \frac{\gamma(i + a)}{\sigma(\gamma - a)}\right].$$

If $(1 - x)\sigma < \gamma$ then $W$ is maximized at $i > i^*$; else it is maximized at $i^*$.

References


